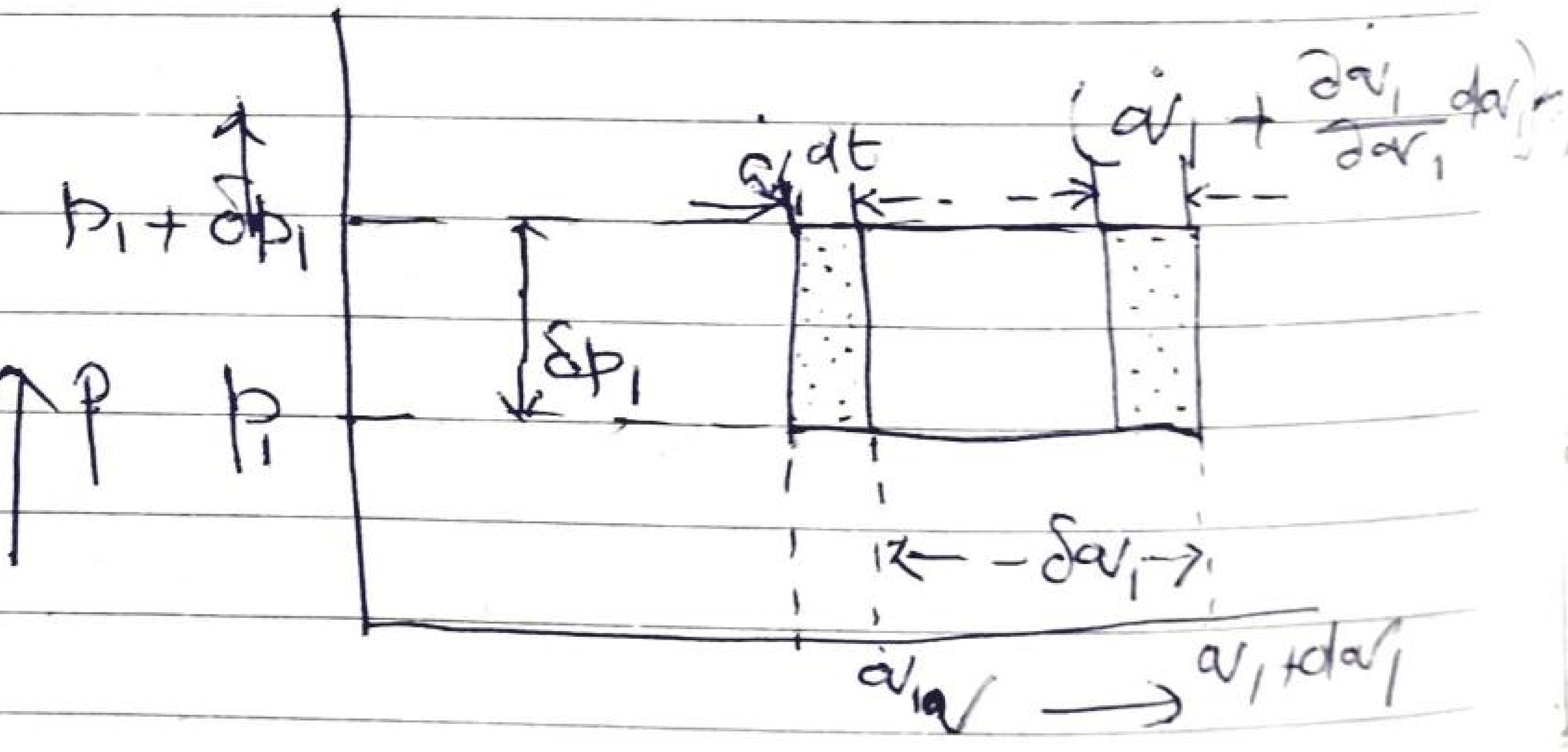
Liouville's Theorem →

Liouville's theorem is known as the principle of conservation of the distribution function in phase-space. This theorem is primarily concerned with defining a fundamental properties of the phase space - the space of position and momenta coordinates, in which the system, represented by a point moves in time. The theorem consists of two parts

(I) The first part states the conservation of density in phase space i.e $d\rho/dt = 0$

(II) The second part gives the conservation of extension in phase space i.e $\frac{d}{dt}(\delta v) = 0$ or the volume at the disposal of a particular number of phase points is conserved throughout the phase space.

First part →



Consider any fixed element of volume of phase space located between q_1 and $q_1 + dq_1$ and p_1 & $p_1 + \delta p_1$.

The number of systems located in this volume ($\delta q_1 \ldots \delta q_f$, $\delta p_1 \ldots \delta p_f$) changes as the coordinates and momenta of the systems vary.

In a time dt the change in number of systems within this volume of phase-space is $(\partial \rho/\partial t)\, dt\, (\delta q_1 \dots \delta q_f)$

when ρ is the density of system

This change is due to the number of systems entering and leaving this volume in time dt.

consider two faces of hyper volume normal to the q axis with coordinates q_1 and $q_1 + \delta q_1$. Number of phase points entering the first face in time dt will be $\rho \dot{q}_1\, dt\, \delta q_2 \dots \delta q_f\, \delta p_1 \dots \delta p_f$ → (1) where ρ and $\dot{q}_1$ are the density and indicated component of velocity for representative points at $q_1 \dots q_f, p_1 \dots p_f$.

Again the phase points leaving the face $(q_1 + \delta q_1)$ in time dt

$$\left(\rho + \frac{\partial \rho}{\partial q_1}\delta q_1\right)\left(\dot{q}_1 + \frac{\partial \dot{q}_1}{\partial q_1}\delta q_1\right) dt\, \delta q_2\, \delta q_f\, \delta p_1 \dots \delta p_f \rightarrow (2)$$

Neglecting higher order terms

$$\left[\rho \dot{q}_1 + \left(\rho \frac{\partial \dot{q}_1}{\partial q_1} + \dot{q}_1\, \partial \rho/\partial q_1\right)\delta q_1\right] dt\, \delta q_2 \dots \delta q_f\, \partial p_1 \dots \delta p_f \quad - (3)$$

Subtracting eqn (3) from eqn (1)

$$-\left(\rho \frac{\partial \dot{q}_1}{\partial q_1} + \dot{q}_1\, \partial \rho/\partial q_1\right) dt\, \delta q_1 - \delta q_f\, \partial p_1 - \partial p_f$$

similarly for p_1 coordinate

$$-\left(\rho \frac{\partial \dot{p}_1}{\partial p_1} + \dot{p}_1 \frac{\partial \rho}{\partial p_1}\right) dt\, \delta q_1 - \delta q_f\, \delta p_1 - \delta p_f$$

The net increase in time dt of number

of systems in this volume of phase space is then obtained by summing the net numbers of systems entering the volume through all the faces labeled by $q_1 \ldots q_f$ and $p_1 \ldots p_f$. Hence

$$\frac{d}{dt}(\delta N) = -\sum_{j=1}^{f} \left\{ \rho\left(\frac{\partial \dot{q}_1}{\partial q_1} + \frac{\partial \dot{p}_1}{\partial p_1}\right) + \left(\frac{\partial \rho}{\partial q_1}\dot{q}_1 + \frac{\partial \rho}{\partial p_1}\dot{p}_1\right)\right\}$$

Now

$$\frac{d}{dt}(\delta N) = \frac{\partial \rho}{\partial t}\, dt\, \delta q_1 \ldots \delta q_f, \partial p_1 \ldots \partial p_f$$

$$\frac{\partial \rho}{\partial t}\, dt\, \delta q_1 \ldots \delta q_f\, \delta p_1 \ldots \delta p_f = \cancel{-\sum_{j=1}^{f} \rho(}$$

$$= -\sum_{j=1}^{f}\left\{\rho\left(\frac{\partial \dot{q}_1}{\partial q_1} + \frac{\partial \dot{p}_1}{\partial p_1}\right) + \left(\frac{\partial \rho}{\partial q_1}\dot{q}_1 + \frac{\partial \rho}{\partial p_1}\dot{p}_1\right)\right\} dt$$

$$\delta q_1 \ldots \delta q_f\, \delta p_1 \ldots \delta p_f$$

$$\frac{\partial \rho}{\partial t} = -\sum_{j=1}^{f}\left\{\rho\left(\frac{\partial \dot{p}_1}{\partial p_1} + \frac{\partial \dot{q}_1}{\partial q_1}\right) + \left(\frac{\partial \rho}{\partial q_1}\dot{q}_1 + \frac{\partial \rho}{\partial p_1}\dot{p}_1\right)\right\}$$

In accordance with the equations of motion in canonical form

$$\dot{q}_1 = \frac{\partial H}{\partial p_1} \qquad \dot{p}_1 = -\frac{\partial H}{\partial q_1}$$

H = Hamiltonian

$$\frac{\partial \dot{q}_1}{\partial q_1} = -\frac{\partial \dot{p}_1}{\partial p_1} = \frac{\partial^2 H}{\partial q_i\, \partial p_i}$$

since the order of differentiation is immaterial

$$\therefore \sum_{j=1}^{f}\left(\frac{\partial \dot{q}_1}{\partial q_1} + \frac{\partial \dot{p}_1}{\partial p_1}\right) = 0$$

$$\boxed{\text{So}\left(\frac{\partial \rho}{\partial t}\right)_{q,p} = -\sum_{i=1}^{f}\left(\frac{\partial \rho}{\partial q_1}\dot{q}_1 + \frac{\partial \rho}{\partial p_1}\dot{p}_1\right)}$$

This result is known as Liouville's theorem.

This eqn can be written as

$$\left(\frac{\partial \rho}{\partial t}\right)_{q,p} + \sum_i \frac{\partial \rho}{\partial q_i}\frac{dq_i}{dt} + \sum_i \frac{\partial \rho}{\partial p_i}\frac{dp_i}{dt} = 0 \quad -(A)$$

This equation is identical with the equation of continuity in hydrodynamics. If ρ is function of q, p, t and q, p are functions of t, the total differential coefficient of ρ w.r.t. t $-(B)$

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial q}\frac{dq}{dt} + \frac{\partial \rho}{\partial p}\,dp/dt$$

comparing (A) & (B)

$$d\rho/dt = 0$$

In accordance with Gibbs, this form of expression called the principle of the conservation of density in phase space. i.e. the density of a group of points remains constant along their trajectories in the phase space. If at any time the phase points are distributed uniformly in phase space, they will for ever have uniform density.

Second part → The number of system

$$\delta N = \rho\,\delta\gamma$$

Differentiating w.r.to t

$$\frac{d}{dt}(\delta N) = \frac{d\rho}{dt}\delta V + \rho\frac{d}{dt}(\delta V)$$

Since the number of phase points δN in a given region of the phase space must remain fixed, so

$$\frac{d}{dt}(\delta V) = 0$$

$$\therefore \quad \frac{d\rho}{dt}(\delta V) + \rho\frac{d}{dt}(\delta N) = 0$$

$$d\rho/dt = 0$$

$$\rho\frac{d}{dt}(\delta V) = 0$$

since $\rho \neq 0$ hence

$$\boxed{d/dt(\delta V) = 0}$$

This eqn gives the principle of conservations of extension in phase space.